J. Fluid Mech. (1976), vol. 77, part 4, pp. 737–751
Printed in Great Britain

Flow around a thin body moving in shallow water

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(Received 2 March 1976)

Approximations for shallow-water ship waves are sought which are valid near the critical speed $U = (gh)^{\frac{1}{2}}$. For sufficiently thin bodies (struts or ships) the governing equation is dispersive. Simple analytic solutions are given which are valid for all $F^2 \leqslant O(1)$. As the thickness increases, nonlinearity also enters. A soliton solution is discussed which applies to a sharp-nosed half-body at slightly supercritical speed.

1. Introduction

With increasing ocean traffic, ship hydrodynamics in shallow water is now an important practical subject. Apart from wave resistance in the direction of the ship's longitudinal axis, lateral forces and moments are affected by the proximity of the sea bottom. While ships are usually designed for certain cruising speeds in deep water, knowledge of these forces is important for safe *operations* in coastal waters or rivers. Considerable theoretical work exists which omits real-fluid effects and is limited to thin or slender ships. For a thin ship with a symmetrical hull and zero angle of attack, Sretenskii (1936) has extended Michell's linearized theory for an infinite ocean (1898) to rectangular canals of arbitrary finite depth and width. Numerical computation from the Michell–Sretenskii theory is tedious and difficult; only in 1966 did Kirsch publish comprehensive results on wave resistance. From her work it is clear that, as the sea depth decreases, the wave drag reaches a peak value at a subcritical speed $F = U(gh)^{-\frac{1}{2}} < 1$, where U is the ship speed and h the sea depth. As h decreases further the peak sharpens but remains finite and shifts towards the critical speed. Thus the neighbourhood of the critical speed should be of considerable theoretical interest and also of practical significance for high-speed boats. A more direct shallow-water approximation based on linearized theory (also due to Michell) is available for a thin strut and was shown by Tuck (1966) to be almost directly applicable in the far field of a symmetrical slender ship moving along its longitudinal axis. This approximation gives reasonable results for Froude numbers not close to unity but is singular and therefore invalid near $F = 1$. It is desirable to find a remedy for this defect since a direct shallow-water approximation should be much more convenient than taking the limit of a theory for arbitrary depth. A useful first step has been undertaken by Lea & Feldman (1972), who modified Tuck's perturbation analysis for the neighbourhood of $F^2 = 1$ and obtained the transonic equation in aerodynamics. By numerical means they calculated the sinkage and trim, which were related to the vertical force and the pitching moment, and obtained finite and continuous results near $F = 1$. No results on the wave drag were given. Their theory points out the importance of nonlinear effects, which should be included for many types of ships in practice. However, it leaves open the theoretical questions whether by accounting for dispersion the linearized shallow-water theory can be made to give a finite limit near $F = 1$ as does the linearized arbitrary-depth theory, and if so what the conditions are for dispersion to be more (or less) important than nonlinearity.

It is well known for stationary shallow water that a long-wave theory which accounts for both nonlinearity and dispersion has the widest range of validity. By applying a Galilean transformation to the Boussinesq equations, Karpman (1967) obtained such equations for ship waves. His attention was focused still on $F^2 > 1$ with $F^2-1 = O(1)$, for which he reduced the governing equation to that of Korteweg & de Vries (KdV).

In this paper we also start with the Boussinesq equations in a moving co-ordinate system, but study more comprehensively several possible approximate regimes. In general, the form of the approximate equation and the extent of its spatial region of uniform validity will be found to depend on the Froude number and the thickness of the strut. We first focus on the neighbourhood of the critical speed $F^2 = 1$. Denoting by B and L the maximum beam and length of the strut and assuming that

$$B/L,\, h/L \ll 1, \tag{1.1}$$

we shall find that a very general equation satisfied by the depth-averaged potential is, in physical variables,

$$(1-F^2)\,\phi_{xx}+\phi_{yy}-\frac{3U}{gh}\,\phi_x\,\phi_{xx}+\frac{h^2}{3}F^2\phi_{xxxx} \cong 0. \tag{1.2}$$

For a very thin strut, $B/h \ll (h/L)^2$, the nonlinear (third) term is negligible and the governing linear equation is that of Rayleigh for the lateral vibration of a beam under an axial load ($F^2 > 1$ for axial tension and $F^2 < 1$ for axial compression). For a rather thick strut, $B/h \gg (h/L)^2$, the dispersive (fourth) term is negligible and the equation reduces to that of transonic gas flow. For $B/h = O(h/L)^2$, however, both nonlinearity and dispersion are important. Despite the premise of $F^2 \approx 1$, it is clear that, if $F^2-1 = O(1)$, (1.2) is dominated by the first two terms, i.e. Michell's approximation

$$(1-F^2)\,\phi_{xx}+\phi_{yy} \cong 0 \tag{1.3}$$

holds. Thus (1.2) is in fact uniformly valid for all $F^2 \leqslant O(1)$ to leading order as long as (1.1) is satisfied. In the remainder of this paper exact solutions will be obtained for the linearized regime, for which very simple results on wave forces will be given for a strut with thickness or at an angle of attack. Evidence of uniformity for all F^2 will be pointed out. For the fully nonlinear dispersive equation a soliton solution will also be discussed.† Most of these results can be

† For $F^2 > 1$, (1.2) is the same as the continuum model of the Fermi–Pasta–Ulam equation for lattice vibrations.

easily reinterpreted for a thin ship with a draft less than the water depth. The purely nonlinear and practically important regime, being essentially the same as transonic aerodynamics, will not be discussed here.

2. Shallow-water approximations and the associated strut thickness

We begin with the exact equations for the inviscid irrotational flow and sketch a direct derivation of the approximate equations. Let ϕ^* be the disturbance potential, ζ^* be the free-surface displacement, (x^*, y^*, z^*) be rectilinear co-ordinates with z^* vertical, and h the uniform still water depth. Let a be the characteristic wave amplitude, which will be later related to the slenderness of the strut, and introduce the following normalized variables:

$$\zeta^* = a\zeta, \quad \phi^* = (gaL/U)\,\phi, \quad x^* = Lx, \quad y^* = Ly, \quad z^* = hz. \tag{2.1}$$

The exact governing equations are

$$\phi_{zz} + \mu^2(\phi_{xx} + \phi_{yy}) = 0 \quad \text{for} \quad -1 < z < \epsilon\zeta(x,y), \tag{2.2a}$$

$$\left.\begin{aligned} F^2\mu^2(\zeta + \phi_x) + \tfrac{1}{2}\epsilon[\mu^2(\phi_x^2 + \phi_y^2) + \phi_z^2] &= 0 \quad (2.2b)\\ \phi_z = \mu^2(F^2 + \epsilon\phi_x)\,\zeta_x + \epsilon\mu^2\phi_y\,\zeta_y & \quad (2.2c)\end{aligned}\right\} \quad \text{at} \quad z = \epsilon\zeta,$$

$$\phi_z = 0 \quad \text{at} \quad z = -1, \tag{2.2d}$$

where $\epsilon \equiv a/h$ and $\mu \equiv h/L$ are both small numbers. Note that the scale for ϕ^* is so chosen that the linear terms in (2.2 b, c) are of leading order. Introducing the same expansion as Rayleigh,

$$\phi = \phi_0 - \frac{\mu^2}{2!}(z+1)^2\,\Delta\phi_0 + \frac{\mu^4}{4!}(z+1)^4\,\Delta\Delta\phi_0 + \dots, \tag{2.3}$$

which satisfies (2.2 a, d) exactly, where $\Delta = \partial_x^2 + \partial_y^2$, we obtain the following approximations by including the leading-order terms, $O(\epsilon)$ and $O(\mu^2)$:

$$\zeta + \phi_{0x} \cong \tfrac{1}{2}\mu^2\Delta\phi_{0x} - (\epsilon/2F^2)\,(\phi_{0x}^2 + \phi_{0y}^2), \tag{2.4}$$

$$F^2\zeta_x + \Delta\phi_0 \cong -\epsilon\zeta\Delta\phi_0 - \epsilon(\phi_{0x}\zeta_x + \phi_{0y}\zeta_y) + \tfrac{1}{6}\mu^2\Delta\Delta\phi_0. \tag{2.5}$$

The terms omitted are $O(\epsilon^2, \epsilon\mu^2, \mu^4)$. To the same accuracy ζ may be eliminated from the above, resulting in

$$\Delta\phi_0 - F^2\phi_{0xx} - \epsilon[\tfrac{1}{2}(\phi_{0x}^2 + \phi_{0y}^2)_x + \nabla\cdot(\phi_{0x}\nabla\phi_0)] + \mu^2[\tfrac{1}{2}F^2\Delta\phi_{0xx} - \tfrac{1}{6}\Delta\Delta\phi_0] \cong 0, \tag{2.6}$$

which has been obtained by Karpman (1967), who applied a Galilean transformation to the Boussinesq equations for waves in stationary shallow water. It is possible to replace ϕ_0 by the depth-averaged potential $\bar{\phi}$ defined by

$$(1+\epsilon\zeta)\,\bar{\phi} = \int_{-1}^{\epsilon\zeta} \phi\,dz. \tag{2.7}$$

From (2.3) we obtain

$$\bar{\phi} = \phi_0 - \tfrac{1}{6}\mu^2\Delta\phi_0 + \dots, \tag{2.8}$$

with which (2.6) may be rewritten as

$$\Delta\bar{\phi}-F^2\bar{\phi}_{xx}-\epsilon[\tfrac{1}{2}(\bar{\phi}_x^2+\bar{\phi}_y^2)_x+\nabla.(\bar{\phi}_x\nabla\bar{\phi})]+\tfrac{1}{3}\mu^2F^2\Delta\bar{\phi}_{xx}\cong 0. \tag{2.9}$$

Equations (2.6) and (2.9) are valid for all $F^2 = O(1)$. For supercritical speeds with $F^2-1 = O(1)$ Karpman (1967) has shown that, for $\epsilon = O(\mu^2)$, one can get a uniformly valid approximation for $O(y) = O(\mu^{-2})$, $y > 0$, by letting

$$\sigma = x-(F^2-1)^{\frac{1}{2}}y, \quad \tau = \mu^2 y, \quad u(\sigma,\tau) = \phi_{0\sigma}, \tag{2.10}$$

which reduces (2.6) to the KdV equation

$$u_\tau+\frac{\epsilon}{\mu^2}\frac{2+F^2}{2(F^2-1)^{\frac{1}{2}}}uu_\sigma-\frac{1}{6}\frac{F^4}{(F^2-1)^{\frac{1}{2}}}u_{\sigma\sigma\sigma}\cong 0, \quad \tau > 0. \tag{2.11}$$

A similar equation can be obtained for $y < 0$.

What is the order of magnitude of a for a given body? This must be found from the boundary condition on the hull. Consider first a strut with zero angle of attack. In dimensionless form, the boundary condition on the body is

$$\frac{\partial\phi}{\partial y}=\frac{B}{h}\frac{\mu}{\epsilon}(F^2+\epsilon\phi_x)\,Y_x^{\pm} \quad \text{on} \quad y=\frac{B}{h}\mu Y^{\pm}(x), \tag{2.12}$$

where $Y^* \equiv BY$, or alternatively,

$$\frac{\partial\phi}{\partial\tau}=\frac{B}{h}\frac{1}{\epsilon\mu}(F^2+\epsilon\phi_x)\,Y_x^{\pm} \quad \text{on} \quad \tau=\frac{B}{h}\frac{1}{\mu}Y^{\pm}(x). \tag{2.13}$$

Consider $\epsilon \gtreqless O(\mu^2)$. For (2.9) to be valid for $(x,y) = O(1)$ we require from (2.12) that

$$\frac{B}{h}\frac{\mu}{\epsilon} = O(1) \quad \text{or} \quad \frac{B}{h}\gtreqless O\left(\frac{h}{L}\right). \tag{2.14}$$

For F^2 not too close to 1, Michell's approximation of keeping the first two terms in (2.6) or (2.9) is correct to leading order. In the second approximation one must account for nonlinearity if $B/h > O(h/L)$, for dispersion if $B/h < O(h/L)$ and for both if $B/h = O(h/L)$. Consider $\epsilon = O(\mu^2)$ again. For (2.11) to hold for $x = O(1)$ and $y = O(\mu^{-2})$ we require from (2.13) that

$$\frac{B}{h}\frac{1}{\epsilon\mu} = O(1) \quad \text{or} \quad \frac{B}{h} = O\left(\frac{h}{L}\right)^3, \tag{2.15}$$

which implies a very thin strut. Karpman appears to be in error in stating that $B/h \gg 1$ instead. For this problem the boundary condition (2.13) leads to an initial condition for u on $\tau = 0$ and the inverse scattering method can be applied in principle for the exact solution. In practice numerical techniques may be more straightforward and will not be discussed here.

As was shown by Tuck, to leading order the boundary-value problem for a strut is equivalent to the outer problem for a slender ship with draft less than h, as long as one replaces $Y^*(x^*)$ by $S^*(x^*)/2h$, where $S^*(x^*)$ is the cross-sectional area of the ship. Furthermore, the pressure on the ship's hull is essentially the pressure of the outer solution measured at $y = 0$. Thus the strut solution may be used to give all the important results for a slender ship with only minor modifications.

More important modifications are, however, needed for a yawed ship with a keel very close to the bottom (Newman 1969).

We now turn to near-critical speeds.

3. Approximate equations near the critical speed $F^2 = 1$

Beginning with (2.9), which is valid for all $F^2 = O(1)$, and the boundary condition (2.13), we consider three cases: (1) $\epsilon > O(\mu^2)$, (2) $\epsilon < O(\mu^2)$ and (3) $\epsilon = O(\mu^2)$.

Case 1. *Large amplitude waves or a 'thick' strut*: $\epsilon > O(\mu^2)$

As is well known in transonic gasdynamics, it is necessary to keep the term $\overline{\phi}_{yy}$ in (2.9) for uniform validity. Thus we introduce

$$\eta_1 = \epsilon^{\frac{1}{2}} y. \tag{3.1}$$

This means that near the critical speed transverse variations become appreciable only far away from the hull. If we also let

$$F^2 - 1 = \tfrac{1}{3}\alpha_1^2 \epsilon \quad \text{with} \quad \alpha_1^2 = O(1) \tag{3.2}$$

equation (2.9) becomes

$$\overline{\phi}_{\eta_1\eta_1} - (\tfrac{1}{3}\alpha_1^2 + 3\overline{\phi}_x)\,\overline{\phi}_{xx} = O(\mu^2/\epsilon). \tag{3.3}$$

Defining $\psi = \frac{1}{9}\alpha_1^2 x + \overline{\phi}$, the above equation becomes

$$\psi_{\eta_1\eta_1} - 3\psi_x \psi_{xx} \cong 0, \tag{3.4}$$

which is identical to the transonic equation and has been derived by Lea & Feldman (1972). Changing y to η_1 in (2.12) we obtain

$$\overline{\phi}_{\eta_1} = \frac{B}{h}\frac{\mu}{\epsilon^{\frac{3}{2}}}(F^2 + \epsilon\overline{\phi}_x)\,Y_x^{\pm} \quad \text{on} \quad \eta_1 = \frac{B}{h}\mu\epsilon^{\frac{1}{2}}Y^{\pm}(x). \tag{3.5}$$

Since $\overline{\phi}_{\eta_1}$ must be $O(1)$ we assume without loss of generality that

$$\frac{B}{h}\frac{\mu}{\epsilon^{\frac{3}{2}}} = 1, \quad \text{which defines} \quad \epsilon = \frac{a}{h} = \left(\frac{B}{L}\right)^{\frac{2}{3}}. \tag{3.6a, b}$$

The approximate boundary condition on the hull is

$$\overline{\phi}_{\eta_1} = F^2 Y_x^{\pm} \quad \text{on} \quad \eta_1 = \epsilon^2 Y^{\pm}(x) \cong \pm 0. \tag{3.7}$$

Since $\epsilon \gg \mu^2$, it follows from (3.6*a*) that the strut must be relatively thick with $B/h \gg \mu^2 = (h/L)^2$. For example $\epsilon = O(\mu^{\frac{4}{3}})$ implies $B/h = O(1)$, which is a very practical range for many types of ships such as aircraft carriers and passenger and cargo ships which can attain near-critical speeds. Much that is known in transonic aerodynamics should therefore be applicable to ships. Proper caution is needed however when there are regions of rapid variations in the solution, so that L is no longer the suitable scale of horizontal length.

Case 2. Infinitesimal amplitudes or a thin strut: $\epsilon \ll \mu^2$

For the same reason as that behind (3.1), we must introduce

$$\eta = \mu y \tag{3.8}$$

to preserve the term $\overline{\phi}_{yy}$ in (2.9). Letting

$$F^2 - 1 = \tfrac{1}{3}\alpha^2\mu^2 \quad \text{with} \quad \alpha = O(1) \tag{3.9}$$

one gets from (2.9) that

$$\overline{\phi}_{\eta\eta} + \tfrac{1}{3}(-\alpha^2\overline{\phi}_{xx} + \overline{\phi}_{xxxx}) = O(\epsilon/\mu^2). \tag{3.10}$$

Substituting (3.8) and (3.9) into the boundary condition (2.12), we may now take

$$B/h = \epsilon, \quad \text{implying} \quad a = B. \tag{3.11}$$

The boundary condition can now be approximated by

$$\phi_\eta = F^2 Y_x^\pm \quad \text{on} \quad \eta = \pm 0. \tag{3.12}$$

The condition $\epsilon \ll \mu^2$ implies that $B/h \ll (h/L)^2$; the hull must be very thin indeed. Equation (3.10) also governs the lateral vibration of an elastic beam under an axial load (compression for $\alpha^2 < 0$ and tension for $\alpha^2 > 0$); the term $\overline{\phi}_{xxxx}$ represents the effect of dispersion. Its simple form now permits a thorough analytical study which is very cumbersome according to Michell–Sretenskii theory for arbitrary depth for a thin ship at zero angle of attack, and quite impossible for a yawed ship. We remark that Michell–Sretenskii theory is also based on linearized conditions on the free surface and on the hull as in (3.12). Hence its limit for shallow water near $F^2 = 1$ should coincide with (3.10) and be subject to the same restriction on hull thickness.

Case 3. Medium amplitude: $\epsilon = O(\mu^2)$

Equations (3.8), (3.9) and (3.11) still apply but the approximate equation is now

$$\overline{\phi}_{\eta\eta} + \tfrac{1}{3}(-\alpha^2\overline{\phi}_{xx} + \overline{\phi}_{xxxx}) - \frac{\epsilon}{\mu^2} 3\overline{\phi}_x \overline{\phi}_{xx} \simeq 0. \tag{3.13}$$

The implied hull thickness is such that

$$B/h = O[(h/L)^2]. \tag{3.14}$$

Now (3.13) contains the dispersion term of (3.10) and the nonlinear term of (3.3). Hence it is the most general, and (3.14) may be taken to mean more liberally that BL^2/h^3 is arbitrary. Furthermore its generality exceeds the neighbourhood of $F^2 = 1$; by letting $\alpha \to \infty$ such that $\alpha\mu$ becomes $O(1)$, both the nonlinear and the dispersive terms become unimportant, so that Michell's approximation suffices in the region $(x, y) = O(1)$ to leading order.

In summary, (3.13) is uniformly valid for all $F^2 \leqslant O(1)$ with the following qualifications. Away from $F^2 = 1$ it reduces to Michell's equation for $B/h \gg (h/L)^2$ and $(x, y) = O(1)$. Near the critical speed it reduces to Rayleigh's beam equation for $B/h \ll (h/L)^2$, $x = O(1)$ and $y = O(\mu^{-1})$ and to the transonic equation if

$B/h \gg (h/L)^2$, $x = O(1)$ and $y = O(\epsilon^{-\frac{1}{2}})$. We further speculate that even in the last category dispersion may need to be considered whenever nonlinearity is strong enough to produce steep gradients anywhere. In comparison with the original equation (2.9), (3.13) is much simpler and should be easier for numerical computations.

Finally, we must impose the boundary condition that there is no disturbance far upstream. We now turn to some explicit solutions.

4. Thin strut or ship at zero angle of attack

We consider (3.10) with the boundary condition on a symmetrical strut:

$$\bar{\phi}_\eta = \begin{Bmatrix} \pm F^2 Y'(x), & |x| < \frac{1}{2}, \\ 0, & |x| > \frac{1}{2}, \end{Bmatrix} \quad \eta = 0. \tag{4.1}$$

Furthermore there should be no disturbance at $x \sim -\infty$. The boundary-value problem can be easily solved by exponential Fourier transformation. We give only the results for $\bar{\phi}_x$, which is directly related to the free-surface height and the pressure.

For supercritical speeds $F^2 \geqslant 1$ or $\alpha^2 \geqslant 0$ we have

$$\bar{\phi}_x(x,\eta) = \frac{3^{\frac{1}{2}}}{4\pi} F^2 \int_{-\infty}^{\infty} dk \frac{-\tilde{Y}_x}{(k^2+\alpha^2)^{\frac{1}{2}}} \exp\{i[kx - k\eta(k^2+\alpha^2)^{\frac{1}{2}}/3^{\frac{1}{2}}]\}, \tag{4.2}$$

where $\tilde{f}(k)$ denotes the Fourier transform of $f(x)$. This solution holds for the side $\eta > 0$; that for the side $\eta < 0$ can be inferred by symmetry. The square root $(k^2+\alpha^2)^{\frac{1}{2}}$ is defined in the complex k plane with two branch cuts along the imaginary axis from $i\alpha$ to $i\infty$ and from $-i\alpha$ to $-i\infty$. The branch is so chosen that $(k^2+\alpha^2)^{\frac{1}{2}}$ is positive for all real k, which ensures that there is no disturbance far upstream.

For subcritical speeds $F^2 \leqslant 1$ or $\alpha^2 = -\beta^2 \leqslant 0$ the solution is

$$\bar{\phi}_x(x,\eta) = \frac{3^{\frac{1}{2}}}{4\pi} F^2 \left\{ \int_{-\beta}^{\beta} dk \frac{ik}{|k|} \frac{-\tilde{Y}_x}{(\beta^2-k^2)^{\frac{1}{2}}} \exp\{[ikx - |k|\,\eta(\beta^2-k^2)^{\frac{1}{2}}/3^{\frac{1}{2}}]\} \right.$$
$$\left. + \left(\int_{-\infty}^{-\beta} + \int_{\beta}^{\infty}\right) dk \frac{-\tilde{Y}_x}{(k^2-\beta^2)^{\frac{1}{2}}} \exp\{ik[x - \eta(k^2-\beta^2)^{\frac{1}{2}}/3^{\frac{1}{2}}]\} \right\}. \tag{4.3}$$

The square root $(k^2-\beta^2)^{\frac{1}{2}}$ is defined in the complex k plane with two cuts parallel to the imaginary axis, one from β to $\beta + i\infty$ and one from $-\beta$ to $-\beta - i\infty$. The branch is so chosen that $(k^2-\beta^2)^{\frac{1}{2}} > 0$ for k real and $|k| > \beta$. Note that at the critical speed $\alpha = \beta = 0$; (4.2) and (4.3) agree since $(k^2+\alpha^2)^{\frac{1}{2}}$ and $(k^2-\beta^2)^{\frac{1}{2}}$ become $|k|$.

Many aspects such as wave patterns can be studied from the simple solution (4.2) and (4.3). We shall however limit our attention to the wave forces.

General formulae for forces on a thin strut or a thin ship

The hydrodynamic pressure on the hull is simply given by

$$p^* \simeq -\rho U \phi^*_{x^*}(x^*, 0), \quad \text{or} \quad p = -\bar{\phi}_x(x, 0), \tag{4.4}$$

with $p = p^*/\rho g B$. Use has been made of (3.11), which implies that B is also the characteristic wave amplitude. The total drag on the hull is

$$R^* = (\rho g B^2 h) R, \quad \text{where} \quad R = -2\int_{-\infty}^{\infty} \bar{\phi}_x Y'\,dx \quad \text{(strut)}. \tag{4.5}$$

As mentioned before, the pressure as given by (4.4) applies also near a ship with its keel above the sea bottom. If the cross-sectional area of the ship is $S^*(x^*) = S_0 S(x)$ we can replace $2Y$ by S and B by S_0/h to obtain the drag

$$R = -\int_{-\infty}^{\infty} \bar{\phi}_x S'\,dx \quad \text{(ship)}. \tag{4.6}$$

Following the argument of Tuck, the vertical force is

$$Z^* = (\rho g B^2 L) Z, \quad Z = \int_{-\infty}^{\infty} -\bar{\phi}_x b\,dx \quad \text{(ship)}, \tag{4.7}$$

where $b = B^*(x^*)/B$ is the dimensionless beam, and the trim moment about the y axis is

$$M_y^* = (\rho g B^2 L^2) M_y, \quad M_y = -\int_{-\infty}^{\infty} x b \bar{\phi}_x\,dx \quad \text{(ship)}. \tag{4.8}$$

For more explicit formulae for a ship we use (4.2) and (4.3) to obtain

$$\bar{\phi}_x(x,0) = F^2 \frac{3^{\frac{1}{2}}}{4\pi}\int_{-\infty}^{\infty} dk \frac{-\tilde{S}_x}{(k^2+\alpha^2)^{\frac{1}{2}}} e^{ikx}, \quad \alpha^2 > 0,$$

$$\bar{\phi}_x(x,0) = F^2 \frac{3^{\frac{1}{2}}}{4\pi}\left\{\int_{-\beta}^{\beta} dk\left(\frac{-ik}{k}\right)\frac{\tilde{S}_x}{(\beta^2-k^2)^{\frac{1}{2}}} e^{ikx} + \left(\int_{-\infty}^{-\beta} + \int_{\beta}^{\infty}\right) dk \frac{-\tilde{S}_x}{(k^2-\beta^2)^{\frac{1}{2}}} e^{ikx}\right\}, \quad \alpha^2 = -\beta^2 < 0.$$

For supercritical speeds, since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{(k^2-\alpha^2)^{\frac{1}{2}}} = \frac{1}{\pi} K_0(\alpha|x|)$$

it follows by the convolution theorem that

$$\bar{\phi}_x(x,0) = -F^2 \frac{3^{\frac{1}{2}}}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, S'(\xi)\, K_0(\alpha|x-\xi|), \quad S'(x) \equiv S_x(x).$$

Hence

$$R(\alpha) = F^2 \frac{3^{\frac{1}{2}}}{2\pi}\iint_{-\frac{1}{2}}^{\frac{1}{2}} dx\, d\xi\, S'(x)\, S'(\xi)\, K_0(\alpha|x-\xi|), \quad F^2 \geqslant 1. \tag{4.9}$$

The limit of critical speed $\alpha \to 0$ is easily obtained by using

$$K_0(\alpha|x-\xi|) \simeq -\ln \alpha|x-\xi|.$$

Assuming $S = 0$ at two ends of the ship we get

$$R(0) = -\frac{3^{\frac{1}{2}}}{2\pi}\iint_{-\frac{1}{2}}^{\frac{1}{2}} dx\, d\xi\, S'(x)\, S'(\xi) \ln|x-\xi|, \tag{4.10}$$

which resembles the Kármán formula for supersonic drag on a slender body, where S' is replaced by S''.

For subcritical speeds

$$R = \int_{-\infty}^{\infty} dx\, S'(x) \left(\frac{3^{\frac{1}{2}}}{2} F^2\right) \frac{1}{2\pi} \left\{ \int_{-\beta}^{\beta} dk \frac{ik}{|k|} \frac{\tilde{S}_x}{(\beta^2 - k^2)^{\frac{1}{2}}} e^{ikx} + \left(\int_{-\infty}^{-\beta} + \int_{\beta}^{\infty} \right) dk \frac{\tilde{S}_x e^{ikx}}{(k^2 - \beta^2)^{\frac{1}{2}}} \right\} = \frac{3^{\frac{1}{2}}}{2} \frac{F^2}{2\pi} \left\{ \int_{-\beta}^{+\beta} dk \frac{ik}{|k|} \frac{|\tilde{S}_x|^2}{(\beta^2 - k^2)^{\frac{1}{2}}} + \left(\int_{-\infty}^{-\beta} + \int_{\beta}^{\infty} \right) dk \frac{|\tilde{S}_x|^2}{(k^2 - \beta^2)^{\frac{1}{2}}} \right\}.$$

Since it is always possible to split $S'(x)$ into two parts which are even and odd in x, we may write $\tilde{S}_x = \tilde{S}_x^e + i\tilde{S}_x^o$, where $\tilde{S}_x^e$ is even and $\tilde{S}_x^o$ odd in k. It follows that $|\tilde{S}_x|^2$ is always even in k. Thus the first integral above vanishes and

$$\begin{aligned} R &= \frac{3^{\frac{1}{2}}}{2\pi} F^2 \int_{\beta}^{\infty} \frac{|\tilde{S}_x|^2\, dk}{(k^2 - \beta^2)^{\frac{1}{2}}} \\ &= \frac{3^{\frac{1}{2}}}{2\pi} F^2 \int_{\beta}^{\infty} \frac{dk}{(k^2 - \beta^2)^{\frac{1}{2}}} \iint_{-\infty}^{\infty} dx\, d\xi\, S'(x)\, S'(\xi)\, e^{ik(x-\xi)} \\ &= \frac{3^{\frac{1}{2}}}{2\pi} F^2 \int_{\beta}^{\infty} \frac{dk}{(k^2 - \beta^2)^{\frac{1}{2}}} \cos k(x-\xi) \iint_{-\infty}^{\infty} dx\, d\xi\, S'(x)\, S'(\xi). \end{aligned}$$

In changing from the second to the third integral, we have dropped a term whose integrand contains a factor $\sin k(x-\xi)$, which is odd about the line $x = \xi$ in the x, ξ plane. Finally, the k integral can be evaluated to give

$$R = -F^2 \frac{3^{\frac{1}{2}}}{4} \iint_{-\frac{1}{2}}^{\frac{1}{2}} dx\, d\xi\, S'(x)\, S'(\xi)\, Y_0(\beta\, |x-\xi|), \quad F^2 \leqslant 1. \tag{4.11}$$

Note that the limit of critical speed (4.10) may be recovered by replacing Y_0 with $(2/\pi) \ln (\beta\, |x-\xi|)$ for small β. This is evidence that the curve of wave drag *vs.* F is continuous at $F = 1$.

Similarly, the vertical lift and trim moments are easily obtained:

$$\begin{pmatrix} Z \\ M \end{pmatrix} = \frac{3^{\frac{1}{2}}}{2\pi} F^2 \iint_{-\frac{1}{2}}^{\frac{1}{2}} dx\, d\xi \begin{pmatrix} b(x) \\ xb(x) \end{pmatrix} S'(\xi)\, K_0(\alpha\, |x-\xi|), \quad F^2 \geqslant 1, \quad \alpha^2 \geqslant 0, \tag{4.12}$$

$$\begin{pmatrix} Z \\ M \end{pmatrix} = \frac{3^{\frac{1}{2}}}{4} F^2 \iint_{-\frac{1}{2}}^{\frac{1}{2}} dx\, d\xi \begin{pmatrix} b(x) \\ xb(x) \end{pmatrix} S'(\xi)\, [\mathbf{H}_0(\beta(x-\xi)) - Y_0(\beta\, |x-\xi|)], \quad F^2 \leqslant 1, \quad \alpha^2 = -\beta^2 \leqslant 0, \tag{4.13}$$

where $\mathbf{H}_0(z)$ is the Struve function.

In order to give further credence to the earlier claim of uniform validity for all $F^2 \leqslant O(1)$, we wish to show that (4.9) reduces to Michell's drag formula for high supercritical speeds $\alpha \to \infty$, and that (4.11) reduces to zero for low subcritical speeds $\beta \to \infty$.

Introducing

$$x + \xi = \mu, \quad x - \xi = \nu, \tag{4.14}$$

we may rewrite (4.9) as

$$R(\alpha) = \frac{3^{\frac{1}{2}} F^2}{2\pi} \left\{ \int_0^1 d\mu \int_0^{1-\mu} d\nu\, S'\left(\frac{\mu+\nu}{2}\right) S'\left(\frac{\mu-\nu}{2}\right) K_0(\alpha\nu) + \int_{-1}^{0} d\mu \int_0^{1+\mu} d\nu\, S'\left(\frac{\mu+\nu}{2}\right) S'\left(\frac{\mu-\nu}{2}\right) K_0(\alpha\nu) \right\}.$$

Since K_0 decays exponentially for large argument, the largest contribution to the ν integral comes from $\nu \approx 0$; we approximate S' by $S'(\frac{1}{2}\mu)$ and approximate the remaining ν integral by noting that

$$\int_0^\infty d\nu\, K_0(\alpha\nu) = \frac{1}{\alpha}\frac{\pi}{2}.$$

Thus to first order we get by using (3.9) that

$$R(\alpha) \cong \frac{hF^2}{2L(F^2-1)^{\frac{1}{2}}}\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\,[S'(x)]^2, \quad F^2-1 > 0, \tag{4.15}$$

which is precisely Michell's formula.

For low subcritical speeds $\beta \to \infty$. Similar transformations via (4.14) and the use of

$$\int_0^\infty Y_0(\nu)\,d\nu = 0$$

lead to the well-known approximation

$$R(\beta) \cong 0, \quad F^2 < 1. \tag{4.16}$$

Similarly, Tuck's formulae for Z and M when F^2 is not close to 1 are limiting cases of (4.12) and (4.13).

Wave drag for a parabolic ship

In order to make a comparison with the numerical results of Kirsch, we choose the same ship, with a rectangular cross-section of draft T and parabolic beam

$$Y^* = \tfrac{1}{2}B[1-(2x^*/L)^2].$$

Thus

$$S^* = BhS(x), \quad \text{where} \quad S = (T/h)(1-4x^2). \tag{4.17}$$

To evaluate the integrals in (4.9), (4.10) or (4.11), we may use (4.14) and get

$$R(\alpha) = F^2\frac{3^{\frac{1}{2}}}{\pi}16\left(\frac{T}{h}\right)^2\int_0^1 d\tau\,(\tfrac{1}{3}-\tau+\tfrac{2}{3}\tau^3)\,K_0(\alpha\tau), \quad \alpha^2 \geqslant 0, \quad \text{supercritical}, \tag{4.18}$$

$$R(\beta) = -F^2\,8\times 3^{\frac{1}{2}}\left(\frac{T}{h}\right)^2\int_0^1 d\nu\,(\tfrac{1}{3}-\tau+\tfrac{2}{3}\tau^3)\,Y_0(\beta\tau), \quad \alpha^2 = -\beta^2 \leqslant 0, \quad \text{subcritical}. \tag{4.19}$$

In particular, at the critical speed either formula above gives

$$R(0) = \frac{2\times 3^{\frac{1}{2}}}{\pi}\left(\frac{T}{h}\right)^2, \quad R^*(0) = \rho g B^2 h\left(\frac{T}{h}\right)^2\frac{2\times 3^{\frac{1}{2}}}{\pi}. \tag{4.20}$$

Among the figures for many size ratios plotted by Kirsch, her figure 3, which is for $L/h = 15$, $B/T = 3$ and $h/T = 2$, is relevant for shallow water and is chosen for comparison. In figure 1 we present the results computed from (4.18) and (4.19), which can be expressed in terms of tabulated functions. Kirsch's results by a much more elaborate computation are indistinguishable from ours except for high supercritical Froude numbers. We note that on this plot of R *vs.* F the

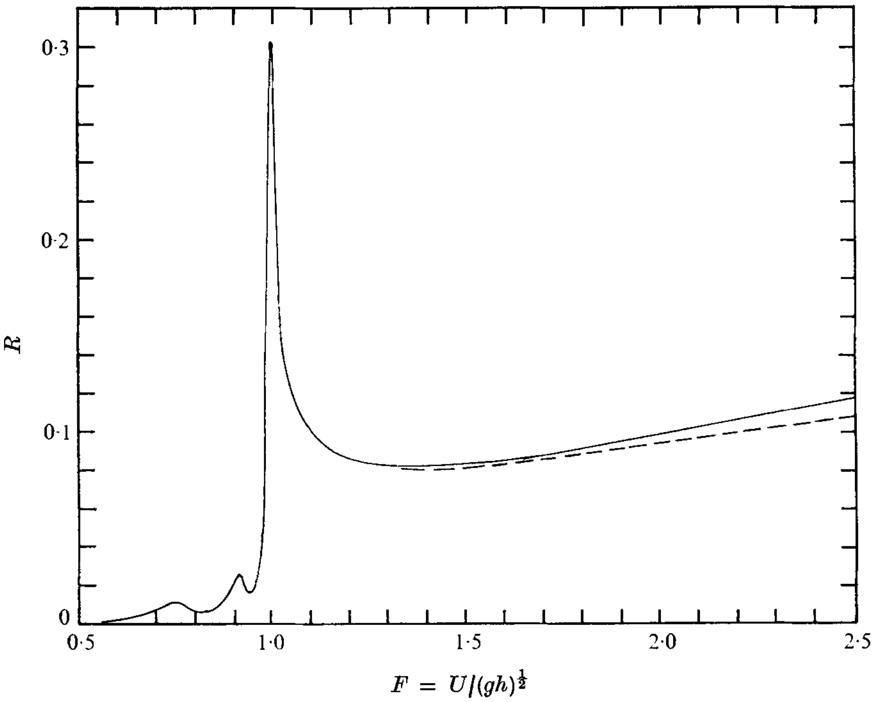

FIGURE 1. Wave drag. ——, present theory; – – –, Kirsch.

maximum drag occurs for a Froude number slightly less than unity ($F \simeq 0{\cdot}996$ or $\beta \simeq 2{\cdot}20$). This has been checked by formally expanding

$$\frac{dR_-}{dF} = \frac{dR_-}{d\beta}\frac{d\beta}{dF} \simeq \frac{-3}{\mu^2\beta}\frac{dR_-}{d\beta} \tag{4.21}$$

for 'small' β and then equating it to zero. Indeed from the general formula (4.11) the same qualitative result can be obtained.

Ship of minimum wave drag

Knowing that in general the maximum drag occurs approximately at the critical speed, let us search for the optimum profile at $F^2 = 1$ with the constraint that the total volume is fixed, the length being already fixed by normalization. Extremizing $R(0)$ from (4.10) with the constraint that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} S\,dx = V \equiv V^*/BhL = \text{constant}, \tag{4.22}$$

we obtain the Euler equation

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} S'(\xi)\ln|x-\xi|\,d\xi = \lambda x,$$

where λ is the Lagrange multiplier. The integral equation can be solved to give

$$S = \tfrac{1}{2}\lambda[(\tfrac{1}{2})^2 - x^2]^{\frac{1}{2}}.$$

Invoking (4.22) we find $\lambda = (16/\pi)\, V$, so that

$$S(x) = (8V/\pi)\,[(\tfrac{1}{2})^2 - x^2]^{\frac{1}{2}}. \tag{4.23}$$

Thus the optimal cross-sectional shape is an ellipse, in contrast to a parabola for high supercritical speeds (Zhukovskii; see Kostyukov 1968, p. 345). Substituting (4.23) into (4.10) we find the minimum drag at the critical speed to be

$$R_m(0) = 4 \times 3^{\frac{1}{2}}\pi^{-1}V^2. \tag{4.24}$$

For a ship of constant draft T and elliptic planform with maximum beam B_e the volume is $V^* = \frac{1}{4}\pi B_e TL$, or $V = \frac{1}{4}\pi T/h$, so that

$$R_m^*(0) = \rho g B_e^2 (T^2/h)\,(\tfrac{1}{4}\pi \times 3^{\frac{1}{2}}).$$

As a check and comparison, we note that a parabolic ship [cf. (4.17)] with equal draft, length and volume must have the beam equal to $\frac{3}{8}\pi B_e$. It follows from (4.18) that the wave drag is $\frac{9}{8}R_m^*(0)$.

5. Slightly yawed plate of zero thickness at critical speed

Consider a thin plate tilted counterclockwise with respect to the oncoming stream at a small angle δ. The characteristic beam scale B may be taken as δL. In dimensionless variables the governing equation (3.10) must be supplemented by the boundary condition that $\overline{\phi}_\eta = 1$ on $\eta = \pm 0$. By symmetry $\overline{\phi}$ must be odd in η, so that it is zero for $\eta = 0$, $|x| > \frac{1}{2}$ and discontinuous, $\overline{\phi}(x, 0+) = -\overline{\phi}(x, 0-)$, for $|x| < \frac{1}{2}$. Again $\overline{\phi} \to 0$ as $x \to -\infty$.

The corresponding problem has apparently not been solved analytically in the context of Michell's thin-ship approximation for arbitrary h. It is a relatively simple exercise in the linear slender-body theory for shallow water for F not close to 1. We shall now give a linearized solution for $F^2 = 1$.

Taking the exponential Fourier transform and setting

$$\overline{\phi}(x, 0+) = \phi_0(x), \quad \text{say}, \tag{5.1}$$

we find the formal solution to be

$$\overline{\phi}(x,\eta) = \frac{1}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, \phi_0(\xi) \int_{-\infty}^{\infty} dk \exp\{ik[(x-\xi) \mp |k|\,\eta/3^{\frac{1}{2}}]\}. \tag{5.2}$$

Applying the boundary condition on the upper side of the plate we have

$$\begin{aligned}\overline{\phi}(x, 0+) = 1 &= \frac{1}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, \phi_0(\xi) \int_{-\infty}^{\infty} \frac{dk}{3^{\frac{1}{2}}} ik\,|k|\, e^{ik(x-\xi)} \\ &= -\frac{1}{3^{\frac{1}{2}}}\frac{\partial^2}{\partial x^2}\left[\frac{1}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, \phi_0(\xi) \int_0^{\infty} \sin k(x-\xi)\, dk\right] \\ &= \frac{1}{3^{\frac{1}{2}}}\frac{\partial^2}{\partial x^2}\frac{1}{\pi} \fint_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d\xi\, \phi_0(\xi)}{\xi - x}, \quad |x| < \tfrac{1}{2}.\end{aligned} \tag{5.3}$$

The last (Cauchy principal) integral is evaluated by noting that

$$\int_0^{\infty} \sin k(x-\xi)\, dk = \lim_{\epsilon \to 0+} \operatorname{Im} \int_0^{\infty} \exp[-\epsilon k + ik(x-\xi)]\, dk.$$

Integrating twice with respect to x, we obtain

$$\frac{1}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d\xi\,\phi_0(\xi)}{\xi - x} = \frac{3^{\frac{1}{2}}}{2}x^2 + C_1 x + C_2, \tag{5.4}$$

which is the classical airfoil equation, C_1 and C_2 being constants of integration. The solution is simply (see, for example, Tricomi 1957, pp. 173–180)

$$\phi_0(x) = -[(\tfrac{1}{2})^2 - x^2]^{-\frac{1}{2}}[\tfrac{1}{2}\times 3^{\frac{1}{2}}(\tfrac{1}{8}x - x^3) + C_1(\tfrac{1}{8} - x^2) - C_2 x + C_3], \tag{5.5}$$

where C_3 is also a constant.

To determine the constants C_1, C_2 and C_3 we require that ϕ_0 is finite at both ends $x = \pm\frac{1}{2}$ and that $\partial\phi_0/\partial x$ is finite at the trailing edge $x = \frac{1}{2}$. These are reasonable conditions to be satisfied in Michell's approximation also. Finally,

$$\phi_0(x) = \tfrac{1}{2}\times 3^{\frac{1}{2}}[(\tfrac{1}{2})^2 - x^2]^{\frac{1}{2}}(\tfrac{1}{2} - x), \quad |x| < \tfrac{1}{2}. \tag{5.6}$$

The pressure and the velocity on the upper side of the plate are therefore

$$p(x, 0+) = -\phi_x(x, 0+) = -\phi_{0x} = \frac{3^{\frac{1}{2}}}{4}\left(\frac{\frac{1}{2} - x}{\frac{1}{2} + x}\right)^{\frac{1}{2}}, \quad |x| < \tfrac{1}{2}, \tag{5.7}$$

which has a square-root singularity at the leading edge and satisfies Kutta's condition at the trailing edge. Equations (5.6) and (5.7) show that the variations of $\phi_0(x)$ and $p(x, 0\pm)$ are the same as those on a classical plate airfoil in low subsonic flow. This is perhaps surprising because the governing equations are so different. The overall flow fields of the two cases, of course, bear little resemblance to each other. The total lift and moment about the origin are easily obtained:

$$\left.\begin{aligned} Y &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx[p(x, 0-) - p(x, 0+)] = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, 2\phi_0(x) = -\frac{3^{\frac{1}{2}}}{4}\pi, \\ \text{or} \qquad Y^* &= -\tfrac{3}{4}\rho g\delta L^2, \end{aligned}\right\} \tag{5.8}$$

$$M_z = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\,(x2\phi_{0x}) = \frac{3^{\frac{1}{2}}}{16}\pi, \quad \text{or} \quad M_z^* = \frac{3^{\frac{1}{2}}\pi}{16}\rho g\delta L^3. \tag{5.9}$$

The lift is negative while the moment is positive because the yaw is anticlockwise. The centre of force is at the quarter-chord point.

The problem with $F^2 \neq 1$ can be treated similarly. The integral equation is Cauchy-singular and can be reduced to a Fredholm integral equation which must be solved numerically or approximately. We forgo this analysis, which does not involve any difficulty in principle.

Finally, the case at a thin plate (ship) with a draft less than h can be treated without difficulty in principle. If the clearance beneath the keel is $O(h)$, the cross-flow can be easily obtained as in usual aerodynamic problems. If the clearance is very small, Newman's analysis (1969) must be applied near the body.

6. A soliton solution

An exact solution to the full nonlinear and dispersive equation (3.13) is possible. Setting

$$\tfrac{1}{3}\alpha^2 = C^2 = (F^2 - 1)/\mu^2 \tag{6.1}$$

and seeking a permanent-wave solution

$$\phi = \phi(\xi), \quad \text{where} \quad \xi = x - K\eta, \tag{6.2}$$

it is easily found by standard arguments that

$$\psi \equiv -\phi' = \frac{C^2-K^2}{\epsilon/\mu^2}\,\mathrm{sech}^2\{\tfrac{1}{2}[3(C^2-K^2)]^{\frac{1}{2}}(x-K\eta)\} \tag{6.3}$$

provided that $C^2 > K^2$, for which it is necessary that the ambient speed is supercritical. We remark that the same solution can be obtained from (2.9) but is different from a similar solution to (2.11) by Karpman.

To find the corresponding strut shape we apply the boundary condition

$$\phi_\eta(x,0) = -K\phi_\xi(\xi = x) = K\psi(x) = Y'. \tag{6.4}$$

Integrating from $-\infty$ to x we obtain

$$Y = \frac{2K}{3^{\frac{1}{2}}}\frac{\mu^2}{\epsilon}(C^2-K^2)^{\frac{1}{2}}\{\tanh \tfrac{1}{2}[3(C^2-K^2)]^{\frac{1}{2}}x+1\},$$

which represents a half-body with an infinitely long sharp nose extending to $x \sim -\infty$ and a constant thickness as $x \sim +\infty$. Since $\zeta \simeq -\overline{\phi}_x = -\overline{\phi}'(\xi)$, the free surface forms two solitary humps stretching out symmetrically on each side of the body, with constant displacement along the lines of equal phase:

$$x \pm K\eta = x \pm K\mu y = \text{constant}, \quad y \gtrless 0.$$

Now the value of K is determined by noting that L should represent the characteristic length of the solitary wave and B the total thickness (i.e. $2Y^*(\infty)$) of the strut. Thus we may require that $C^2 - K^2 = 1$,

so that

$$K = \left(\frac{F^2-1}{(h/L)^2} - 1\right)^{\frac{1}{2}} \tag{6.5}$$

and

$$\frac{4K}{3^{\frac{1}{2}}}\frac{\mu^2}{\epsilon}(C^2-K^2)^{\frac{1}{2}} = \tfrac{1}{2}. \tag{6.6}$$

Consequently, we have

$$Y = \frac{1}{2}\left(\tanh\frac{3^{\frac{1}{2}}}{2}x+1\right), \quad Y' = \frac{3^{\frac{1}{2}}}{4}\mathrm{sech}^2\frac{3^{\frac{1}{2}}}{2}x \tag{6.7}$$

and

$$\psi = \frac{K}{2}\mathrm{sech}^2\frac{3^{\frac{1}{2}}}{2}(x-K\mu y), \quad y > 0. \tag{6.8}$$

The phase lines are nearly normal to the x axis for $F^2 \simeq 1$ and slant in the downstream direction for increasing F^2. Upon substituting (6.5) into (6.6), K can be eliminated to give

$$\frac{3^{\frac{1}{2}}}{8}\frac{B}{h} = \left(\frac{h}{L}\right)\left[F^2-1-\left(\frac{h}{L}\right)^2\right]^{\frac{1}{2}}. \tag{6.9}$$

Thus, for a fixed Froude number, B and L must be related by (6.9) to give rise to a solitary wave on each side of the body. In particular, the smallest Froude number is

$$\min F^2 = 1 + (h/L)^2, \tag{6.10}$$

at which the half-body has no thickness.

The dimensionless wave drag is

$$R = -2\int_{-\infty}^{\infty} \bar{\phi}_x Y' \, dx = 2\int_{-\infty}^{\infty} \psi Y' \, dx = K\frac{3^{\frac{1}{2}}}{4}\int_{-\infty}^{\infty} \operatorname{sech}^4 \frac{3^{\frac{1}{2}}}{2} x \, dx$$

$$= \frac{2K}{3} = \frac{2}{3}\left(\frac{F^2-1}{(h/L)^2} - 1\right)^{\frac{1}{2}}. \tag{6.11}$$

Apart from its theoretical interest, the value of this exact solution is to provide a check for future numerical solutions for more realistic body shapes. Cnoidal wave and n-soliton solutions can also be obtained but they correspond to strut profiles of much less interest to ship hydrodynamics.

This research has been financially supported by a grant from the U.S. National Science Foundation (ENG 74-02576). The author is indebted to his colleague Dick K. P. Yue for computing the curve in figure 1.

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